# Mergers, Welfare, and Concentration: Results from a Model of Stackelberg-Cournot Oligopoly

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*This paper explores the relationship between mergers, welfare, and concentration, using a two-stage oligopoly model that generalizes the Cournot and Stackelberg models. This model has been used to show that some profitable mergers raise welfare and that some welfare-lowering mergers are unprofitable. Based on this, one might conclude that policy designed to restrict mergers is unnecessary or even counterproductive. This present paper examines in greater detail the implications of this model and finds that a merger's effects depend not only on the reduction in the number of firms, but also on premerger and postmerger firm behavior. In fact, most mergers lower welfare, and many of these are profitable. Usually, but not always, changes in concentration and welfare are negatively related. (JEL D43, L12, L13)* *Atlantic Econ. J., 29(4): pp. 378-392, Dec. 01.* 

## Introduction

The effect of mergers on market outcomes is the subject of some controversy. Among most noneconomists, the prevailing folk wisdom is that restraints on horizontal mergers are necessary because mergers result in increased industry concentration, higher prices, and lower social welfare. This mode of thought underlies antitrust policy in most countries. However, the theoretical industrial organization literature is replete with examples which suggest, in many situations, that mergers are either unprofitable for the merging firms (so that they would not be expected to occur) or they are welfare raising. Within Cournot's [1838] simple model of oligopoly, for example, mergers between firms always reduce social welfare. However, as has been shown [Salant et al., 1983], in nearly all cases, there is no incentive for firms to merge because the profits of the merging firms decrease as a result of the merger (except when the merger results in a monopoly). Daughety [1990] looks at the effects of a merger within a more general two-stage oligopoly model and finds that for a particular class of mergers, if a merger is welfare raising, it is profitable for the merging firms.

Based on such results, neither author concludes that horizontal restraints are completely unnecessary. Salant et al. [1983] do not argue against merger policy at all. Rather, they use their results to critique the use of Cournot's model for analyzing mergers. At most, Daughety argues that a policy of always opposing mergers is misguided. However, based on their examples, one might mistakenly get the impression that a policy that ever opposes horizontal mergers is at best inconsequential and at worst deleterious. The goal of this paper is to put the findings they report into some sort of perspective by further examining Daughety's model

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(which includes Cournot's simple model as a special case). All possible two-firm mergers within this model are considered, and their effects on the profitability of the merging firms, social welfare, and industry concentration are examined. While some mergers are indeed unprofitable and others welfare raising, there remain many mergers that are both profitable and welfare lowering, thus reaching the unsurprising conclusion that a role remains for merger policy. In particular, Daughety's result obviously still holds that if a certain type of merger is welfare raising, it is profitable for the merging firms. However, all mergers of this type are profitable for the merging firms, not merely the welfare-raising ones. Another type of merger that is possible within this framework is always profitable for the merging firms and always welfare lowering. Also, concentration, as measured by the Herfindahl index, is a useful, though imperfect, guide to assessing the welfare effects of a merger. Finally, this paper discusses the implications of these results for possible extensions of the model that allow for endogenously chosen mergers.

## The Stackelberg-Cournot Model

The model here is that of Daughety, which is a simplification of a more general two-stage oligopoly model [Sherali, 1984].[1] An industry is made up of $N \geq 1$ firms producing a homogeneous product. Of these firms, $n_l \geq 0$ are leader firms, and $n_f = N - n_l \geq 0$ are follower firms. The sequence of firms' decisions is as follows. First, all leaders simultaneously choose their output levels, then all followers observe the leader's choices before simultaneously choosing their own output levels. Thus, leaders have Cournot conjectures regarding the other leaders, and followers have Cournot conjectures regarding the other followers, but both leaders and followers know that followers can (and do) react to the quantity choices of the leaders.

The market price is determined by the linear inverse demand curve $p = a - bQ$, where $Q$ is industry output and $a$ and $b$ are positive parameters. All firms face the same constant average cost, $c \in [0, a)$.[2] Define $Q_l$ to be the combined output of all leader firms and $Q_f$ is the combined output of all follower firms so that $Q = Q_l + Q_f$. The values of $a, b, c, n_l$, and $n_f$ are exogenous and common knowledge among the firms, as is the sequence of moves.

This model generalizes several other models of oligopoly. Setting either $n_f$ or $n_l$ to zero, with the other strictly positive, reduces the model to the standard Cournot model. The version of the model with one leader and one follower $(n_l = n_f = 1)$ is equivalent to the standard Stackelberg [1934] model. Allowing more than one follower $(n_l = 1, n_f \geq 1)$ results in a special case of Robson's [1990] model.

This game always has a unique subgame perfect equilibrium outcome, which was derived by Daughety. In the present study:

$$q_f(n_l, n_f) = \frac{a-c}{b}\left[\frac{1}{(n_l+1)(n_f+1)}\right] \quad \text{and} \quad q_l(n_l, n_f) = \frac{a-c}{b}\left[\frac{1}{n_l+1}\right] .$$

This outcome will be referred to as the Stackelberg-Cournot outcome. Note that the output levels of the leaders do not depend on the number of followers, and followers produce only $1/(n_f + 1)$ times as much as leaders. Thus, the equilibrium market quantity and price are:

$$Q(n_l, n_f) = \frac{a - c}{b}\left[1 - \frac{1}{(n_l + 1)(n_f + 1)}\right] , \tag{1}$$

and

$$p(n_l, n_f) = c + \frac{a - c}{(n_l + 1)(n_f + 1)} . \tag{2}$$

Each leader earns profit:

$$\pi_l(n_l, n_f) = \frac{(a - c)^2}{b(n_l + 1)^2 (n_f + 1)} , \tag{3}$$

and each follower earns:

$$\pi_f(n_l, n_f) = \frac{(a - c)^2}{b(n_l + 1)^2 (n_f + 1)^2} . \tag{4}$$

So industry profits are:

$$\begin{aligned} \Pi(n_l, n_f) &= n_l\pi_l + n_f\pi_f \\ &= \frac{(a - c)^2}{b}\left[\frac{(n_l + 1)(n_f + 1) - 1}{(n_l + 1)^2 (n_f + 1)^2}\right] . \end{aligned} \tag{5}$$

Since there are no fixed costs, industry profits are equal to producer surplus. Consumer surplus is:

$$\begin{aligned} S_c(n_l, n_f) &= \frac{1}{2}Q(a - p) \\ &= \frac{(a - c)^2}{2b}\left[1 - \frac{1}{(n_l + 1)(n_f + 1)}\right]^2 , \end{aligned} \tag{6}$$

and total welfare (consumer surplus plus industry profits) is:

$$W(n_l, n_f) = \frac{(a - c)^2}{2b}\left[1 - \frac{1}{(n_l + 1)^2 (n_f + 1)^2}\right] , \tag{7}$$

so that deadweight loss is:

$$\frac{(a - c)^2}{2b}\left[\frac{1}{(n_l + 1)^2 (n_f + 1)^2}\right] .$$

Before moving on to mergers, the following result is introduced, showing that when the number of either type of firm changes, holding the other constant, the result corresponds closely to that of a change in the number of firms in the standard Cournot model.[3]

Lemma 1: For any $n_l$ and $n_f$, an increase (decrease) in either $n_l$ or $n_f$, leaving the other constant, results in an increase (decrease) in quantity, consumer surplus, and total welfare, and a decrease (increase) in price and industry profits.

Proof: See the Appendix.

## Mergers and Welfare

The standard merger result from Cournot oligopoly with linear demand and constant average cost [Salant et al., 1983] is that two-firm horizontal mergers are unprofitable unless the merging firms were the only firms in the industry. The effects of a merger in the Stackelberg-Cournot model will naturally depend on the types (leader or follower) of the merging firms and the type of firm they become after the merger. The cases in this model that correspond most closely to mergers in Cournot oligopoly are those when two leader firms merge to form a single leader or when two follower firms merge to form a single follower. The profitability (or lack thereof) of such mergers and their effect on welfare are similar to their counterparts in the Cournot model.

Result 1: A merger of two leaders to form a single leader, or two followers to form a single follower, is profitable if and only if the merging firms were the ony firms of their type. Should such a merger occur, price and total profits would increase, and quantity, consumer surplus, and total welfare would decrease.

Proof: See the Appendix.

Thus, just as Salant et al. [1983] established for the standard Cournot model, a merger between two like firms to produce a firm of the same type is indeed welfare lowering but nearly always unprofitable for the merging firms. This is true whether the merger involves two followers forming a follower or two leaders forming a leader. The difference between this result and its analog in the standard Cournot model is that here, the merging firms do not need to be the only two firms in the industry, merely the only firms of their type in the industry.

However, if two followers merge to form a leader, the result is less obvious. The decrease in the number of firms will tend to reduce welfare, but since the number of leaders increases and leaders produce a higher output level than followers, there is a countervailing tendency toward higher quantity and, thus, higher welfare. Previously, it has been argued [Hart, 1975] that mergers of small firms might actually increase competition and be welfare raising. Daughety states, and it is easily verified, that a merger of two followers to form a leader is welfare raising if the industry is initially made up predominantly of followers. In the present study, if $n_f > 2n_l + 3$, market quantity and welfare are increasing by the merger. If $n_f < 2n_l + 3$, quantity and welfare are decreased by the merger. If $n_f = 2n_l + 3$, quantity and welfare are unchanged by the merger. Daughety also states that such a merger, if welfare raising, is profitable for the merging firms.[4] It will now be seen that not only is such a merger profitable when it is welfare raising, it is nearly always profitable for the merging firms.

Result 2: A merger of two followers to form a single leader is never unprofitable to the merging firms. If $n_l = 0$ and $n_f = 3$, such a merger results in no net change in profits for the merging firms. In all other cases, it is strictly profitable.

Proof: See the Appendix.

In other words, welfare-raising mergers of this type are always profitable, but so are welfare-lowering mergers. Thus, knowing that such a merger is profitable (and likely to occur) does not say whether or not it will improve welfare.

The final case considered is the merger of a leader and a follower. It seems reasonable to assume that the resulting merged firm is a leader.[5]

Result 3: A merger of one leader and one follower to form a leader is always profitable to the merging firms. Following such a merger, price and total profits increase, and quantity, consumer surplus, and total welfare decrease.

Proof: See the Appendix.

This is similar to the case of two like firms merging to form a firm of the same type in that welfare unambiguously decreases. In both cases, the effect on welfare is unsurprising since the effect on industry structure is simply a decrease in the number of one type of firm with no change in the other type. The difference between these two cases is that unlike when two like firms merge to form a firm of the same type, a merger between leader and follower is always profitable for the merging firms and therefore would be expected to take place in an industry unfettered by merger regulations. The case of two followers merging to form a leader is more complicated because the numbers of both types of firms change.

## Mergers and Concentration

This paper now examines the effect on concentration of the various two-firm mergers that are possible within this model. The Herfindahl index ($H$) will be used to measure concentration [Hirschman, 1945; Herfindahl, 1950].[6] In the Stackelberg-Cournot outcome:

$$\begin{aligned} H(n_l, n_f) &= n_f\left(\frac{q_f}{Q}\right)^2 + n_l\left(\frac{q_l}{Q}\right)^2 \\ &= \frac{n_f + n_l(n_f+1)^2}{(n_l n_f + n_l + n_f)^2} \quad . \end{aligned} \tag{8}$$

A change in either the number of either leaders or the number of followers, with the other constant, normally has the expected effect—concentration increases as in the standard Cournot model—though in one special case, the opposite happens.

Lemma 2: A decrease of 1 in $n_f$ increases $H$, for all $n_l$ and $n_f$. A decrease of 1 in $n_l$ decreases $H$ if $n_l = 1$ and $n_f > 2$. Otherwise, it increases $H$.

Proof: See the Appendix.

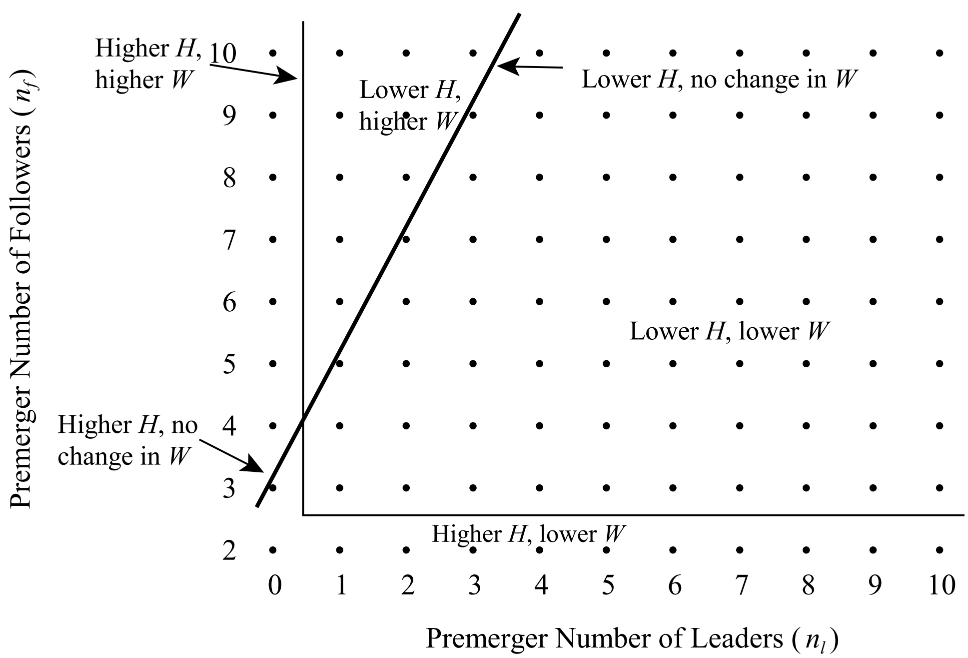
Given Lemma 2, the effects on concentration of most of the mergers considered here can be easily determined. Most, though not all, raise concentration.

Result 4: Mergers of two like firms to form a firm of the same type and mergers between a leader and a follower to form a leader result in an increase in concentration.

Proof: The effect of either of these mergers on industry structure is simply the loss of either one leader or one follower, with no change in the number of firms of the other type. By Lemma 2, such a merger increases *H*. Note that the situation in Lemma 2 where a decrease in the number of leaders reduces *H* does not apply here, as a merger between leaders must begin with at least two leaders.

In these cases, changes in concentration following a merger correspond perfectly to changes in welfare. Irrespective of profitability, these mergers increase *H* and decrease *W*, as expected. This is not necessarily true for the remaining type of merger considered in this paper. The effect on *H* of a merger between two followers to form a leader depends on the initial industry structure. It is possible for such a merger to actually decrease concentration.

**FIGURE 1**
**Effects of Merger Between Two Followers to Form One Leader on Concentration (*H*) and Welfare (*W*)**




Result 5: Mergers of two followers to form a leader result in an increase in concentration in exactly two cases:

1) if the merging firms had been the only two followers in the industry; and
2) if there previously had been no leaders in the industry.

Proof: See the Appendix.

Note that in Case 1, since $n_f = 2$, it must be that $n_f < 2n_l + 3$, so that such a merger reduces welfare. In Case 2, $n_l = 0$, so the merger will lower welfare if $n_f = 2$ (which also falls into the first case), it will leave welfare unchanged if $n_f = 3$, and it will raise welfare if $n_f \geq 4$. Therefore, when examining a merger of two followers to form a leader, the change in concentration is not much help in predicting the change in welfare. Figure 1 depicts the effects of this type of merger on concentration and welfare for values of $n_l$ and $n_f$ up to 10. Note that most reduce both concentration and welfare.

## Discussion

Because it is widely believed outside the economics profession that mergers result in increased concentration and lower social welfare, it is not surprising that a strand of the industrial organization literature has responded by showing that some welfare-lowering mergers are unprofitable for the involved firms (and thus unlikely to happen) and that some profitable mergers are welfare raising. However, by focusing only on such samples, there is a risk in forgetting that many potential mergers are both profitable and welfare lowering. The results of this paper do not contradict those of Daughety, Salant et al., and others, but they serve only to place them into context. Daughety's point that mergers can be welfare raising is well taken. However, many mergers do reduce welfare, and a substantial fraction of these are profitable for the merging firms. Specifically within the model considered here, there are several classes of profitable mergers. A merger between two like firms resulting in a single firm of the same type is profitable if and only if there were initially only two firms of that type. Such a merger always reduces welfare. A merger between a leader and a follower resulting in a leader is always profitable. Such a merger also always reduces welfare. A merger between two followers resulting in a leader is also always profitable (except in the one case where profit is unchanged). Such a merger is welfare lowering unless the industry is originally predominantly made up of followers.

It would be relatively straightforward to imagine an extension of this model in which mergers are endogenous and are undertaken as long as they are profitable. Notice that nearly all profitable mergers reduce $n_f$ and none increases it. As previously mentioned, the only profitable merger that does not reduce $n_f$ is a merger between the only two leaders in the industry. If there are initially many followers and few leaders in an industry, profitable mergers may at first raise welfare, but these mergers would eventually lower $n_f$ to the point where further profitable mergers would only lower welfare. The end result would be an industry made up of only leaders, a relatively low welfare outcome.[7] If the industry were instead made up of mostly leaders, or similar numbers of leaders and followers, the end result would be the same, but the total loss in welfare would be even larger.

Measures of concentration, such as the Herfindahl index, can be useful for assessing the effects of a potential merger, but they are not perfect. When leaders merge, either with followers or other leaders, concentration is perfectly negatively correlated with welfare, as the result is always a decrease in welfare and an increase in concentration.[8] This is also the result when two followers merge into one. However, when followers merge with each other to form a leader, there is little if any relationship between the merger's effect on welfare and its effect on concentration. Welfare-raising mergers may increase or decrease concentration, as may welfare-lowering mergers.

These results highlight the difficulty in establishing a sensible merger policy. A completely *laisser-faire* policy will allow many welfare-lowering mergers to take place. On the other hand, a policy that simple proscribes all mergers will prevent even those mergers that are welfare raising and should be allowed (or even encouraged). Ideally, a merger policy would not only look at the premerger and postmerger number of firms, but also the behavior of firms taking part in a potential profitable merger and their likely behavior afterward, as well as that of the other firms in the industry. Although, assessing which firms are leaders and which are followers is, of course, less straightforward in practice than in this model.

## APPENDIX

Proof of Lemma 1: Consider a change in $n_l$ (similar reasoning will hold in all cases for a change in $n_f$). The derivative of $Q$ with respect to $n_l$ is:

$$\frac{\partial Q}{\partial n_l} = \frac{a-c}{b} \frac{1}{(n_l+1)^2 (n_f+1)} ,$$

which is positive for all nonnegative $n_l$ and $n_f$. Thus, $Q$ increases with $n_l$. Since $p = a - bQ$, $p$ decreases with $Q$ and therefore with $n_l$. Also, because $p = a - bQ$, (6) can be written $S_c = (1/2)bQ^2$. Then, $\partial S_c / \partial Q = bQ$, so consumer surplus increases with $Q$ and therefore with $n_l$. Also, the derivative of welfare with respect to $n_l$ is:

$$\frac{\partial W}{\partial n_l} = \frac{(a-c)^2}{b} \frac{1}{(n_l+1)^3 (n_f+1)^2} ,$$

which is positive for all nonnegative $n_l$ and $n_f$. Thus, $W$ increases with $n_l$. Finally, the derivative of industry profits with respect to $n_l$ is:

$$\frac{\partial \Pi}{\partial n_l} = \frac{(a-c)^2}{b} \frac{1}{(n_l+1)^2 (n_f+1)} ,$$

which is negative for all nonnegative $n_l$ and $n_f$. Thus, $\Pi$ increases with $n_l$.

Proof of Result 1: Consider a merger of two leaders to form one leader. Suppose the industry begins with $n_l \geq 2$ leaders and $n_f$ followers. The effect of such a merger on industry structure is a decrease of 1 in $n_l$ with no change in $n_f$. By (3), the two leaders' total premerger profit is:

$$2\pi_l(n_l, n_f) = 2 \cdot \frac{(a-c)^2}{b(n_l+1)^2 (n_f+1)} ,$$

and the profits of the merged firm afterward are:

$$\pi_l(n_l - 1, n_f) = \frac{(a-c)^2}{bn_l^2(n_f+1)} .$$

The change in profits for the (potentially) merging firms is:

$$\begin{aligned}\pi_l(n_l - 1, n_f) - 2\pi_l(n_l, n_f) &= \frac{(a-c)^2}{b(n_f+1)}\left[\frac{1}{n_l^2} - \frac{2}{(n_l+1)^2}\right] \\ &= \frac{(a-c)^2}{b(n_f+1)} \cdot \frac{-n_l^2 + 2n_l + 1}{n_l^2(n_l+1)^2} .\end{aligned}$$

This expression is positive if and only if $-n_l^2 + 2n_l + 1$ is positive, which occurs if and only if $n_l = 2$, regardless of $n_f$.

Now consider a merger of two followers to form one follower. Suppose the industry begins with $n_l$ leaders and $n_f \geq 2$ followers. The effect of such a merger on industry structure is a decrease of 1 in $n_f$, with no change in $n_l$. By (4), the two followers' total premerger profit is:

$$2\pi_f(n_l, n_f) = 2 \cdot \frac{(a-c)^2}{b(n_l+1)^2(n_f+1)^2} ,$$

and the profits of the merged firm afterward are:

$$\pi_f(n_l, n_f - 1) = \frac{(a-c)^2}{b(n_l+1)^2 n_f^2} .$$

The change in profits for the (potentially) merging firms is:

$$\begin{aligned}\pi_f(n_l, n_f - 1) - 2\pi_f(n_l, n_f) &= \frac{(a-c)^2}{b(n_l+1)^2}\left[\frac{1}{n_f^2} - \frac{2}{(n_f+1)^2}\right] \\ &= \frac{(a-c)^2}{b(n_f+1)^2} \cdot \frac{-n_f^2 + 2n_f + 1}{n_f^2(n_f+1)^2} .\end{aligned}$$

This expression is positive if and only if $-n_f^2 + 2n_f + 1$ is positive, which occurs if and only if $n_f = 2$, regardless of $n_l$. Thus, either type of merger is profitable for the merging firms if and only if the premerger number of firms of their type is exactly 2. Since the effect on industry structure of either of these mergers is simply a loss of either one leader or one follower, by Lemma 1, price and total profits increase, and quantity, consumer surplus, and total welfare decrease.

Proof of Result 2: The effect on the industry of such a merger is a decrease of 2 in $n_f$ and an increase of 1 in $n_l$. So if the industry begins with $n_l \geq 0$ leaders and $n_f \geq 2$ followers, the total profits of the two merging firms before the merger were (by (4)):

$$2\pi_f(n_l, n_f) = 2 \cdot \frac{(a-c)^2}{b(n_l+1)^2(n_f+1)^2} \quad ,$$

and the profits of the merged firm afterward are (by (3)):

$$\pi_l(n_l+1, n_f-2) = \frac{(a-c)^2}{b(n_l+2)^2(n_f-1)} \quad .$$

Therefore, the net change in profits for the merging firms is:

$$\begin{aligned} &\pi_l(n_l+1, n_f-2) - 2\pi_f(n_l, n_f) \\ &= \frac{(a-c)^2}{b(n_l+2)^2(n_f-1)} - \frac{2(a-c)^2}{b(n_l+1)^2(n_f+1)^2} \\ &= \frac{(a-c)^2}{b}\left[\frac{(n_l+1)^2(n_f+1)^2 - 2(n_l+2)^2(n_f-1)}{(n_l+1)^2(n_f+1)^2(n_l+2)^2(n_f-1)}\right] \quad . \end{aligned}$$

It is easy to see that the first coefficient, $(a-c)^2/b$, and the denominator of the second coefficient, $(n_f+1)^2(n_l+1)^2(n_f-1)(n_l+2)^2$, are always positive. The numerator of the second coefficient is $(n_f+1)^2(n_l+1)^2 - 2(n_f-1)(n_l+2)^2$. The merger is profitable if and only if this expression is positive. To simplify the math, the substitutions $x = n_f - 1$ and $y = n_f - 1$ are made. Since $n_l \geq 0$ and $n_f \geq 2$, it must be that $x \geq 1$ and $y \geq 1$. The numerator of the second coefficient can then be written as:

$$\begin{aligned} (y+2)^2x^2 - 2y(x+1)^2 &= (x^2y^2 + 4x^2y + 4x^2) - (2x^2y + 4xy + 2y) \\ &= x^2y^2 + 2x^2y + 4x^2 - 4xy - 2y \\ &= (x^2y - 2)y + 2xy(x-2) + 4x^2 \quad . \end{aligned}$$

If $x \geq 2$, all three of these terms are nonnegative and the first and third are strictly positive. So the entire expression is positive. If $x = 1$, this expression further reduces to $y(y-2) - 2y + 4 = (y-2)^2$, which is zero when $y = 2$ (that is, $n_f = 3$) and positive otherwise. Combining these two cases, it is found that the merger is profitable for all

$n_l$ and $n_f$, except when $n_l = 0$ and $n_f = 3$, in which case, the merger has no effect on the profits of the merging firms.

Proof of Result 3: Suppose the industry begins with $n_l \geq 1$ leaders and $n_f \geq 1$ followers. The effect on the industry of such a merger is a decrease of 1 in $n_f$ and no change in $n_l$. So, if the industry begins with $n_l \geq 1$ leaders and $n_f \geq 1$ followers, by (3) and (4), the two firms' total premerger profit is:

$$\begin{aligned}\pi_l(n_l, n_f) + \pi_f(n_l, n_f) &= \frac{(a-c)^2}{b(n_l+1)^2(n_f+1)} + \frac{(a-c)^2}{b(n_l+1)^2(n_f+1)^2} \\ &= \frac{(a-c)^2(n_f+2)}{b(n_l+1)^2(n_f+1)^2} \quad ,\end{aligned}$$

and the profit of the merged firm afterward is:

$$\pi_l(n_l, n_f - 1) = \frac{(a-c)^2}{b(n_l+1)^2 n_f} \quad .$$

The net change in profits for the merging firms is then:

$$\begin{aligned}\pi_l(n_l, n_f - 1) - \left[\pi_l(n_l, n_f) + \pi_f(n_l, n_f)\right] &= \frac{(a-c)^2}{b(n_l+1)^2 n_f} - \frac{(a-c)^2(n_f+2)}{b(n_l+1)^2(n_f+1)^2} \\ &= \frac{(a-c)^2}{b(n_l+1)^2 n_f(n_f+1)^2} \quad .\end{aligned}$$

This change in profits is positive for any $n_l$ and $n_f$. Since the effect on industry structure of this merger is simply a loss of one follower, by Lemma 1, price and total profits increase, and quantity, consumer surplus, and total welfare decrease.

Proof of Lemma 2: From (8):

$$\begin{aligned}\frac{\partial H}{\partial n_l} &= \frac{(n_l n_f + n_l + n_f)^2(n_f+1)^2 - 2\left[n_f + n_l(n_f+1)^2\right](n_l n_f + n_l + n_f)(n_f+1)}{(n_l n_f + n_l + n_f)^4} \\ &= -\frac{(n_f+1)\left[(n_l - 1)n_f + 2n_l n_f + n_l + n_f\right]}{(n_l n_f + n_l + n_f)^3} < 0 \quad ,\end{aligned}$$

for all $n_f$ whenever $n_l \geq 1$. Now, from (8), $H = (n_f + (n_f+1)^2)/(2n_f+1)^2$ when $n_l = 1$, and $H = 1/n_f$ when $n_l = 0$. So, as $n_l$ increases from zero to 1, the change in $H$ is:

$$H(1,n_f) - H(0,n_f) = -\frac{1}{n_f} + \frac{n_f + (n_f+1)^2}{(2n_f+1)^2} = \frac{n_f^3 - n_f^2 - 3n_f - 1}{n_f(2n_f+1)^2} \quad .$$

The denominator of this expression is always positive. The numerator is positive if $n_f \geq 3$ and negative otherwise. Also:

$$\frac{\partial H}{\partial n_f} = \frac{(n_l n_f + n_l + n_f)^2\left[1 + 2n_l(n_f+1)\right] - 2\left[n_f + n_l(n_f+1)^2\right](n_l n_f + n_l + n_f)(n_l+1)}{(n_l n_f + n_l + n_f)^4}$$

$$= \frac{-3n_l n_f - n_l - n_f}{(n_l n_f + n_l + n_f)^3} < 0 \quad ,$$

for all $n_l$ and $n_f$.

Proof of Result 5: From (8), concentration before the merger is:

$$H(n_l, n_f) = \frac{n_f + n_l(n_f+1)^2}{(n_l n_f + n_l + n_f)^2} \quad ,$$

and after the merger is:

$$H(n_l+1, n_f-2) = \frac{n_f - 2 + (n_l+1)(n_f-1)^2}{\left[(n_l+1)(n_f-2) + n_l + n_f - 1\right]^2} \quad .$$

So the change in concentration is:

$$H(n_l+1, n_f-2) - H(n_l, n_f) = \frac{n_f - 2 + (n_l+1)(n_f-1)^2}{\left[(n_l+1)(n_f-2) + n_l + n_f - 1\right]^2} - \frac{n_f + n_l(n_f+1)^2}{(n_l n_f + n_l + n_f)^2}$$

$$= \frac{(n_l n_f + n_l + n_f)^2\left[n_f - 2 + (n_l+1)(n_f-1)^2\right] - \left[(n_l+1)(n_f-2) + n_l + n_f - 1\right]^2\left[n_f + n_l(n_f+1)^2\right]}{\left[(n_l+1)(n_f-2) + n_l + n_f - 1\right]^2 (n_l n_f + n_l + n_f)^2} \quad .$$

The denominator of this last expression is always positive, so the change in concentration is of the same sign as the numerator. Suppose $n_l > 0$. Then, if $n_f = 2$, the numerator simplifies to:

$$(3n_l + 2)^2(n_l+1) - (n_l+1)^2(9n_l+2) = (n_l+1)(n_l+2) > 0 \quad .$$

So when the merging firms are the only followers in the industry, *H* increases as a result of the merger. If $n_f = 3$, the numerator simplifies to:

$$(4n_l + 3)^2 (4n_l + 5) - (2n_l + 3)^2 (16n_l + 4) = -32n_l^2 - 32n_l + 9 < 0 \quad ,$$

and if $n_f = 4$, the numerator simplifies to:

$$(5n_l + 4)^2 (9n_l + 11) - (3n_l + 5)^2 (25n_l + 4) = -151n_l^2 - 161n_l + 76 < 0 \quad .$$

Finally, if $n_f \geq 5$, the numerator can be written as:

$$\begin{aligned} &-n_l^2 n_f^4 - n_l n_f^4 + 8n_l^2 n_f^2 - 2n_l n_f^3 + n_f^4 - 4n_l^2 n_f + 18n_l n_f^2 - 7n_l^2 - 14n_l n_f + 11n_f^2 - 9n_l - 9n_f \\ &= n_l^2 n_f^2 (25 - n_f^2) + n_f^4 (1 - n_l) + n_l n_f^2 (22 - 17n_l - n_f) + n_f^2 (11 - 5n_f) - n_l n_f^3 \\ &\quad - 4n_l^2 n_f - 4n_l n_f^2 - 5n_f^3 - 14n_l n_f - 9n_l - 9n_f \quad . \end{aligned}$$

The first three addends are nonpositive, the last addend is strictly negative, and all of the subtrahends are strictly positive. So this expression is strictly negative. Thus, if there were initially five or more followers, the merger decreases *H*. Combining these last three cases, it can be seen that if there are initially more than two followers and at least one leader, the merger decreases *H*.

Now suppose $n_l = 0$. Then the numerator of the expression for the change in concentration simplifies to:

$$n_f^2 \left[ n_f - 2 + (n_f - 1)^2 \right] - (2n_f - 3)^2 n_f = n_f^4 - 5n_f^3 + 11n_f^2 - 9n_f \quad ,$$

which can be rewritten as $n_f (n_f - 3)^2 + (n_f - 1)^2 + 4(n_f - 2)$. Inspection verifies that this expression is positive for any $n_f \geq 2$. Thus, when there are initially no leaders in the industry, the merger increases *H*.

## Footnotes

1. Sherali's model has the same structure as the one used here but it allows more general demand and cost curves. He does not look at the effects of mergers but rather proves the existence of the equilibrium under these general conditions.
2. Because average cost is constant in Daughety's model, one source of welfare gains from merger—economies of scale—is not present here [Farrell and Shapiro, 1990].
3 Daughety states this result also and proves it for changes in $n_l$.
4. To be precise, he only looks at the case where the number of firms is, at most, 28. However, as Result 2 shows, his conjecture is true for any number of firms.

5. If the resulting firm is a follower, it can be shown that such a merger always reduces welfare and is only profitable if the merging firms were the only two firms in the industry.
6. This index is commonly used as a measure of industry concentration. For example, it has been shown theoretically [Stigler, 1964] that collusion among firms can be more easily enforced in an industry with high *H*. In addition, many researchers [Cowling and Waterson, 1976] have posited this index as a good indicator of price-cost margin and, thus, social welfare.
7. For example, suppose there were initially 2 leaders and 28 followers. Mergers of two followers to form a leader would be profitable and would at first result in increases in welfare until there were 8 leaders and 16 followers. At this point, the deadweight loss would be only about 32 percent of what it initially was. Then continued mergers of this type (or any other profitable ones) would reduce welfare. The end result would be an industry made up of 16 leaders and no followers, and deadweight loss would be more than 26 times what it initially was.
8. This result is not general. If economies of scale exist, a merger may increase both concentration and welfare [Farrell and Shapiro, 1990].

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